

# 1 Conceptual part of the assignment (Optional and fun)

1. Read extra material under the HW2 tab on the course website, that are excellent if you want a deeper dive into the Lagrangian formalism, principle of least action, euler-lagrange equations, and phonons can be found.
2. What principle/law endows the principle of least action with its validity? Or is it a standalone law unto itself?
3. If you have some time, its worth asking yourself under what conditions would you excite the acoustic or optical mode of the lattice. Which one is more "difficult" to excite? Why is the optical mode called the optical mode? Has it got something to do with light?
4. What if the simple 1d chain of masses came in three varieties, with three different masses? Would you have a third branch? What would the characteristics of the different branches be?
5. An often quoted statement is that the beautiful mathematical structure endowed to conservative systems, a Hamiltonian/Lagrangian etc.etc., is simply not present for dissipative systems. An example is something as simple as a Newtonian fluid. Why is this? This whole range of phenomena comes under the umbrella of non-equilibrium thermodynamics (wikipedia website on this is a good read).
6. For those taking first year graduate asymptotics, you will recollect the section on boundary layers. From a dimensional analysis point of view you will note that you can construct, at minimum, two independent lengthscales: one that arises from the dominant balance in the bulk, and another one from the boundary. Thus, in general, if you can construct multiple timescales or lengthscales given a set of parameters that are deemed to be important then it indicates the presence of multiple time and lengthscales that the system might probe. Of course, as you see in boundary layer theory, the regimes under which the different lengthscales gain important depend on the governing equations and boundary conditions. For example, you don't expect to see a boundary layer in a periodic system without boundaries, right?

## 2 Mathematical part of assignment

### 1. Numerical analysis of normal modes of a simple 1D chain

Starting with the governing equation for the  $s$ th mass  $M \frac{d^2 u_s}{dt^2} = C(u_{s+1} - 2u_s + u_{s-1})$ , and the ansatz  $u_s = q_s e^{i\omega t}$ , you get the equation  $\omega^2 q_s = -C(u_{s+1} - 2u_s + u_{s-1})$ . This can be written as a  $N \times N$  matrix equation for the vector  $\vec{q} = \{q_1, q_2, \dots, q_N\}$  in a system with  $N$  masses of the form  $M\vec{q} = \omega^2 \vec{q}$ . The normal modes analysis we conducted so laboriously in class can be restated as a eigen analysis of the matrix  $M$ .

- What is  $M$  in terms of parameters in the problem?
- Comparing to the normal mode analysis, what is the correct interpretation of the eigenvalues and eigenvectors of the matrix  $M$ ?
- Please calculate and plot the eigenvalues and eigenvectors numerically and interpret in terms of the analytical calculations done in class.

### 2. Long ranged interactions

In consider the simple 1D chain of masses connected with springs we only took into account interactions with nearest neighbors. These are what the springs represented. For a real physical lattice of atoms there are longer range interactions (next nearest, and next next nearest neighbor). These can be modeled "effectively" as springs with different (manifestly weaker) stiffnesses. What would the dispersion relation be for a system with nearest, and next to nearest, neighbor interactions?

### 3. Vibrations of a square lattice

Extend the straightforward 1d calculation of normal modes, and calculate the dispersion relation of a two-dimensional lattice with lattice spacing  $a$  in both directions, and only nearest neighbor interactions modeled as simple springs. After writing the govern equation of motion for each degree of freedom, introduce the ansatz  $u(s, r) = u e^{i(sk_x a + rk_y a - \omega t)}$ , and calculate the  $\omega$  dependence on  $k_x$  and  $k_y$ .

### 4. Dimensional analysis: Please estimate the size of an atom on the basis of dimensional considerations

Suggested steps: 1) What are the two important pieces of physics that determine the size of an atom? What wants to keep electrons close to oppositely charge protons? Why doesn't this attraction lead to an atom of "zero" size where

electrons are on top of protons? This is what is called the uncertainty principle:  $\bar{h} = \Delta x \Delta p$ , which limits how well we can know the location of an electron (if we knew it lay right next to the much more massive atomic nucleus then it would necessarily be whizzing around in a frantic manner – we wouldn't know its momentum). This is essentially what sets the size of an atom. The charge interactions between electrons and protons, and Heisenberg's uncertainty principle. Based on just this you can estimate the size of an atom. What is it?

5. **Dimensional analysis: Heat diffusion** Dimensional analysis can also be used to solve certain types of partial differential equations. If this seems too good to be true, it isn't. Consider the temperature field  $T(x,t)$  at position  $x$  along a 1D bar of metal at time  $t$ . The governing equation of motion turns out to be the diffusion equation  $\frac{\partial T}{\partial t} = D \frac{\partial^2 T}{\partial x^2}$ . Note that this equations look very different from the wave equation  $\frac{\partial^2 T}{\partial t^2} = v^2 \frac{\partial^2 T}{\partial x^2}$  seen earlier in class, where  $v$  is the wave velocity. The solution of the wave equation can be attained using separation of variables. The diffusion equation has a very different structure, which fundamentally is endowed by the random/stochastic nature in which molecules in the bar bump into each other. Consider now the scenario where we inject a certain amount of heat at an arbitrary location, which without loss of generality we can call  $x=0$ , at a single moment in time. We assume that heat is conserved at all times, so any amount injected must remain in the bar, which implies that  $\int_{-\infty}^{+\infty} T(x,t) dx = Q$ . Remember that heat is a form of energy. What's the spatio-temporal evolution of the temperature field?
- What are the units of  $D$ , in the diffusion equation? What are the units of  $Q$ ?
  - We would like to express temperature in the form  $T(x,t) = \mathcal{F}(Q, D, t)\Phi(\xi)$ .  $\mathcal{F}$  is a combination of  $(Q, D, t)$  that has the units of temperature, and  $\Phi$  is a function of a dimensionless combination of  $(x, D, t)$ , defined to be the variable  $\xi$ . In essence what the ansatz enforces is that space and time are not independent variables but that actually only a special combination of  $(x,D,t)$  appear in the evolution of the thermal field. This is often referred to as searching for a similarity solution of the governing Partial Differential Equation (PDE). What is  $\mathcal{F}$  that sets the scale of temperature, and what

is  $\xi$  (the dimensionless combination of  $(x,D,t)$ )?

- (c) Plugging this ansatz into the governing equation should reduce your PDE into an ordinary differential equation (ODE), which are much easier to solve analytically than PDEs.
- (d) Solve the resulting ODE.

## 6. Mutualism

Following the ideas discussed in the class, analyze the solution of the following model of a mutualistic ecology

$$\frac{dN_1}{dt} = r_1 N_1 \left( 1 - \frac{N_1}{K_1} + b_{12} \frac{N_2}{K_1} \right) \quad (1)$$

$$\frac{dN_2}{dt} = r_2 N_2 \left( 1 - \frac{N_2}{K_2} + b_{21} \frac{N_1}{K_2} \right) \quad (2)$$

- (a) After non-dimensionalizing, identify the fixed points of the equations.
- (b) Conduct a stability analysis of the fixed points
- (c) Identify global behavior using nullcline analysis
- (d) Simulate and confirm analytical predictions