PROBLEM SET 4 APPLIED MATHEMATICS 201

Due: November 4

(1) *Final Project* Write a brief summary of the topic for your final project. The summary should explain
  (a) what is the topic you will study,
  (b) what is the scope of work that you will carry out, and
  (c) give 3 references to get you started.

  The topic can be anything that relates to the ideas we've been talking about in class (specifically it can overlap with your current research). We'll give you feedback on the feasibility of your project and potential additional references that might be worth looking at. If you are unsure about what project to pick or about its scope, come and talk to us about it (or email us)

(2) **Problem 2:** *A boundary layer problem*

  In class we described the solution to the boundary layer problem

$$\epsilon y'' + a(x)y' + b(x)y = 0,$$

  with $y(0) = 3$ and $y(1) = -2$.
  (a) Invent two examples of $a(x), b(x)$. For the first example, pick one where the boundary layer will exist on the right side. For the second example pick one where the boundary layer will exist in the middle of the interval.
  (b) Solve your problems numerically.
  (c) Briefly outline the boundary layer solution to your problem, using the method in class as a guide.
  (d) Quantitatively compare your boundary layer solution to the numerical solution, for several values of $\epsilon = 0.1, 10^{-2}, 10^{-3}, \ldots$. Pay special attention to the case where the boundary layer occurs in the middle of the interval. Here please plot the maximum value of your solution ($max_x y(x)$) and compare the magnitude to the prediction from the boundary layer solution.

(3) *Action Potentials* Hodgkin and Huxley won the Nobel Prize for discovering the physiological basis for signal propagation in the brain . At the heart of their analysis was their 1952 solution to a mathematical problem, very similar to the one we have been discussing in class:

  They needed to construct solutions to the following set of differential equations

$$\frac{a}{2RU^2}\frac{d^2V}{dt^2} = C_M\frac{dV}{dt} + g_K n^4(V - V_K) + g_{Na} m^3 h(V - V_{Na} + g_l(V - V_l),$$

where U is the unknown propagation velocity, and the functions n, m, h obey their own differential equations: the equation for n is

$$\frac{dn}{dt} = \alpha_n(1-n) - \beta_n n,$$

with the other equations being listed on page 518 of the attached paper by Hodgkin and Huxley. The coefficients α, β obey complicated nonlinear relations outlined on page 519. Use the parameters outlined by Hodgkin and Huxley in their paper so that you can compare directly to their result.

The goal of this problem is to reproduce Hodgkin and Huxley's analysis for finding a solution to the above equations, and the corresponding value of U . We need a solution where the voltage approaches a constant $V \to constant$ as $t \to \pm\infty$ Hodgkin and Huxley state

> *This is an ordinary differential equation and can be solved numerically, but the procedure is still complicated by the fact that U is not known in advance. It is necessary to guess a value of U , insert it in eqn. (30) and carry out the numerical solution starting from the resting state at the foot of the action potential. It is then found that V goes off towards either $\pm\infty, accordingastheguessedU$ was too small or too large. A new value of 0 is then chosen and the procedure repeated, and so on. The correct value brings V back to zero (the resting condition) when the action potential is over.*

This is much like the logic we described in class, except that there is also an unknown parameter U. This parameter is specified as follows: you will find that in order for V to approach a constant at positive and negative infinity, there will be some number of conditions that need to be imposed. The solution we are looking for is the solution to a second order differential equation and so there are two free parameters. But as for the case in class, the solution is translation invariant. Thus we really only have a one parameter family–just like in class. But when you analyze the number of growing solutions at $\pm\infty$, you will see that it requires two conditions for a solution to be realized. Thus, according to the arguments we gave in class, a generic solution does not exist, at least if U is arbitrary. But if you *allow U to be a free parameter as well*, then there are enough conditions!

On page 524 of the attached paper, Hodgkin and Huxley describe how they find the solution. You should follow their lead (and use the shooting method which is exactly what we outlined in class) but use the modern tools at your disposal to find the answer. Good luck! Their solutions are shown in Fig. 17.