Dimensional analysis: G. I. Taylor and the Trinity explosion

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2007–01–19

During World War II, the British government cooperated with the US on the development of the atomic bomb in the Manhattan project.

G. I. Taylor, a British fluid dynamicist, was asked by his government to study mechanical ways of measuring the bomb's yield (energy output).

Taylor was not directly involved in the bomb's development, and for security reasons worked independent of the US project.

He knew that the energy would be released from a small volume, and would produce a very strong shock wave that would expand in approximately a spherical shape.

He used dimensional analysis to estimate how the radius would scale with the other physical variables.

From his work in fluids, Taylor assumed* the relevant variables would be:

- r, the radius of the shock front.
- ρ, density of surrounding air.
- E, energy released by the device.
- t, the time at which the front reaches r.

*Doing this correctly is the real trick.

These physical variables have dimensions:

$$
\begin{aligned}
r &\sim [L], \\
\rho &\sim [ML^{-3}], \\
E &\sim [ML^2T^{-2}], \\
t &\sim [T].
\end{aligned}
$$

If this is hard to see, fall back on thinking about units.

So assume:

$$r = g(\rho, E, t),$$

where g is some function of powers of the variables. This means that

$$r = C\rho^x E^y t^z,$$

where x, y, z are unknown exponents. The dimensions will satisfy:

$$L = [ML^{-3}]^x [ML^2T^{-2}]^y [T]^z.$$

Expand the exponents:

$$L = M^x L^{-3x} M^y L^{2y} T^{-2y} T^z$$

Now by just looking at this, extract 3 equations for x, y, z:

$$\begin{aligned}
1 &= -3x + 2y \;\text{ from L's,} \\
0 &= x + y \;\text{ from } M\text{'s,} \\
0 &= -2y + z \;\text{ from T's.}
\end{aligned}$$

This is a simple system to solve. You can start from the middle: $y = -x$ and solve it by substitution.

The result is

$$x = -1/5,\ y = 1/5,\ z = 2/5.$$

for the exponents, so our governing equation is

$$r = C\rho^{-1/5}E^{1/5}t^{2/5},$$

where C is some constant that we don't know. We rearrange for energy, first by raising to the fifth power:

$$r^5 = C'E\rho^{-1}t^2$$
$$E = C''\frac{r^5\rho}{t^2}$$

Taylor had experimental data that indicated that $C'' \approx 1.033$ for air.

In 1947 a movie of the Trinity test explosion was released to the public. In one frame $r = 100$ m at a time of $t = 0.016$ s after the explosion. $\rho \approx 1.1$ kg/m^3 at that altitude.

http://nuclearweaponarchive.org/Usa/Tests/Trinity.html

Substitute in these values:

$$E \approx 4 \times 10^{13} \text{ J}.$$

1000 tons of TNT (a kiloton) releases about 4.2×10^{12} J. So the above value is about 10 kilotons TNT equivalent. The actual yield was 18–22 kilotons.

Even closer values can be obtained from other frames. See

http://en.wikipedia.org/wiki/Nuclear_weapon_yield

Not bad for the back-of-the-envelope.*

*Maybe not exactly Taylor's analysis, but close.

Warnings and pitfalls

Remember mathematical functions only take dimensionless arguments. This is shown by power series expansions:

$$
\begin{aligned}
f(\xi) &= e^{\xi} \\
&= 1 + \xi + \frac{1}{2}\xi^2 + \cdots
\end{aligned}
$$

In this case the leading term is obviously dimensionless, and all terms added to it must be also. In general a function has terms of many different orders, which must be dimensionless to add up.

- Some ratios of variables and their derivatives can lead to ambiguous cases. See eq. 2.23:

$$\omega_B = \sqrt{\frac{g}{\theta}\frac{d\theta}{dz}}.$$

  This equation is dimensionally correct for *any* substitution for θ.

- Derivatives and ratios are indistinguishable to a dimensional analysis: g/z has the same dimensions as dg/dz.

- Dimensional analysis is an aid to insight: it cannot completely describe the physics.